

ALMOST HERMITIAN MANIFOLDS WITH VANISHING BOCHNER
CURVATURE TENSOR

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Abstract

We prove that if an almost Hermitian manifold of dimension greater than 4 and vanishing Bochner curvature tensor is not Kähler at a point, then it is flat in a neighbourhood of this point.

Key words: Almost Hermitian manifolds, Kähler manifolds, Bochner curvature tensor.

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1. Introduction. In [1] S. Bochner introduced a curvature tensor as a Kähler analogue of the Weil conformal curvature tensor. Since then there are many results on the Bochner's tensor for Kähler manifolds, as well as for some more general classes of almost Hermitian manifolds, see e.g. [2], [8] and the references in them. These results show that the Bochner tensor is in fact a perfect analogue of the Weil tensor, although we don't know for the moment its geometric meaning.

For the more general classes of almost Hermitian manifolds we can use the classical Bochner tensor, as well as its extension, due to Tricerri and Vanhecke [7]. For some results concerning the last tensor see e.g. [3], [4], [6] and the included references. The main result of [6] has recently been reproved in [5].

The aim of this paper is to show that the vanishing of the classical Bochner tensor provoke very strong restrictions on an almost Hermitian manifold, so it seems more natural to use in this case the tensor of Tricerri-Vanhecke. Namely we prove the following result:

Theorem. *Let M be an $2n$ -dimensional ($n > 2$) almost Hermitian manifold with vanishing classical Bochner curvature tensor. If M is not Kählerian at a point p , then M is flat in a neighbourhood of p .*

2. Preliminaries. Let M be a $2n$ -dimensional almost Hermitian manifold with Riemannian metric g and almost complex structure J . The curvature tensor, the Ricci tensor and the scalar curvature are denoted by R , ρ and τ , respectively. The Bochner

curvature tensor B is defined by $B = R - \varphi(Q)$, where

$$\begin{aligned}\varphi(Q)(x, y, z, u) &= g(x, u)Q(y, z) - g(x, z)Q(y, u) \\ &+ g(y, z)Q(x, u) - g(y, u)Q(x, z) \\ &+ g(x, Ju)Q(y, Jz) - g(x, Jz)Q(y, Ju) - 2g(x, Jy)Q(z, Ju) \\ &+ g(y, Jz)Q(x, Ju) - g(y, Ju)Q(x, Jz) - 2g(z, Ju)Q(x, Jy)\end{aligned}$$

and

$$Q = \frac{1}{2(n+2)}\rho - \frac{\tau}{8(n+1)(n+2)}g .$$

Suppose the Bochner tensor vanishes and let e_1, \dots, e_{2n} be an orthonormal basis of $T_p M$. After adding the identities

$$R(x, e_i, e_i, y) = \varphi(Q)(x, e_i, e_i, y)$$

for $i = 1, \dots, 2n$ we obtain

$$\rho(x, y) = \frac{1}{2(n+2)}\{(2n+1)\rho(x, y) + 3\rho(Jx, Jy)\} .$$

Hence the Ricci tensor ρ is hybrid:

$$\rho(x, y) = \rho(Jx, Jy) .$$

So the tensor Q is also symmetric and hybrid:

$$(2.1) \quad Q(x, y) = Q(y, x) \quad \text{and} \quad Q(x, y) = Q(Jx, Jy) .$$

As a direct consequence of (2.1) M is an AH_1 -manifold, i.e.

$$(2.2) \quad R(x, y, z, u) = R(x, y, Jz, Ju) ,$$

see also [8]. Using (2.2) and the second Bianchi identity

$$(2.3) \quad (\nabla_x R)(y, z, u, v) + (\nabla_y R)(z, x, u, v) + (\nabla_z R)(x, y, u, v) = 0$$

we derive

$$\begin{aligned}(2.4) \quad & R(y, z, (\nabla_x J)u, Jv) + R(y, z, Ju, (\nabla_x J)v) \\ & + R(z, x, (\nabla_y J)u, Jv) + R(z, x, Ju, (\nabla_y J)v) \\ & + R(x, y, (\nabla_z J)u, Jv) + R(x, y, Ju, (\nabla_z J)v) = 0 .\end{aligned}$$

Note also that

$$g((\nabla_x J)y, y) = 0 \quad g((\nabla_x J)y, Jy) = 0 .$$

Although the proof of our theorem can be made using real tangent vectors, we will use the complexification of the tangent spaces. In particular let $\{e_\alpha, Je_\alpha\}$, $\alpha = 1, \dots, n$ be an orthonormal basis of the tangent space $T_p M$ of M at p and denote:

$$Z_\alpha = e_\alpha - iJe_\alpha , \quad Z_{\bar{\alpha}} = e_\alpha + iJe_\alpha .$$

Then

$$(2.5) \quad JZ_\alpha = iZ_\alpha , \quad JZ_{\bar{\alpha}} = -iZ_{\bar{\alpha}} .$$

By (2.1) and (2.5) we find:

$$Q(Z_\alpha, Z_\beta) = 0 \quad Q(Z_\alpha, Z_{\bar{\beta}}) = Q(Z_{\bar{\beta}}, Z_\alpha) .$$

We have also for any Z , α , β

$$g((\nabla_Z J)Z_\alpha, Z_{\bar{\beta}}) = 0 .$$

In the following we assume that the basis $e_\alpha, J e_\alpha$, $\alpha = 1, \dots, n$ diagonalize Q , i.e.

$$Q(e_\alpha, e_\beta) = Q(e_\alpha, J e_\beta) = 0 \quad \text{for} \quad \alpha \neq \beta.$$

Then

$$Q^1(Z_\alpha) = \mu_\alpha Z_\alpha \quad Q^1(J Z_\alpha) = \mu_\alpha J Z_\alpha \quad \alpha = 1, 2, \dots, n$$

for some real numbers μ_α , Q^1 being the tensor of type (1,1), corresponding to Q .

3. Proof of the theorem. It is sufficient to prove, that if M is non Kählerian in p , then it is flat at p .

First we suppose that there exist α, β such that

$$g((\nabla_{Z_{\bar{\beta}}} J) Z_\beta, Z_\alpha) \neq 0.$$

We put in (2.4) $X = Z_\alpha$, $Y = Z_{\bar{\alpha}}$, $Z = Z_{\bar{\beta}}$, $U = Z_\alpha$, $V = Z_\beta$ and we obtain:

$$(3.1) \quad (5\mu_\alpha + \mu_\beta) g((\nabla_{Z_{\bar{\beta}}} J) Z_\beta, Z_\alpha) = 0.$$

With $X = Z_\alpha$, $Y = Z_{\bar{\beta}}$, $Z = Z_{\bar{\gamma}}$, $U = Z_\beta$, $V = Z_\gamma$ for $\gamma \neq \alpha, \beta$ (2.4) implies:

$$(3.2) \quad (\mu_\alpha + \mu_\gamma) g((\nabla_{Z_{\bar{\beta}}} J) Z_\beta, Z_\alpha) + (\mu_\alpha + \mu_\beta) g((\nabla_{Z_{\bar{\gamma}}} J) Z_\gamma, Z_\alpha) = 0.$$

Now let in (2.4) $X = Z_{\bar{\beta}}$, $Y = Z_\gamma$, $Z = Z_{\bar{\gamma}}$, $U = Z_\beta$, $V = Z_\alpha$. The result is:

$$(3.3) \quad (\mu_\alpha + \mu_\beta + 2\mu_\gamma) g((\nabla_{Z_{\bar{\beta}}} J) Z_\beta, Z_\alpha) - (\mu_\beta + \mu_\gamma) g((\nabla_{Z_{\bar{\gamma}}} J) Z_\gamma, Z_\alpha) = 0.$$

Since $g((\nabla_{Z_{\bar{\beta}}} J) Z_\beta, Z_\alpha) \neq 0$ we can derive from (3.1), (3.2) and (3.3) that M is flat at p . Indeed, consider the two possibilities:

Case I: $g((\nabla_{Z_{\bar{\gamma}}} J) Z_\gamma, Z_\alpha) = 0$. Then (3.1), (3.2) and (3.3) imply

$$\begin{cases} 5\mu_\alpha + \mu_\beta = 0 \\ \mu_\alpha + \mu_\gamma = 0 \\ \mu_\alpha + \mu_\beta + 2\mu_\gamma = 0 \end{cases}$$

and hence $\mu_\alpha = \mu_\beta = \mu_\gamma = 0$.

Case II: $g((\nabla_{Z_{\bar{\gamma}}} J) Z_\gamma, Z_\alpha) \neq 0$. By (3.1) we have

$$5\mu_\alpha + \mu_\beta = 0.$$

Changing β and γ we have also

$$5\mu_\alpha + \mu_\gamma = 0$$

and the last two give $\mu_\beta = \mu_\gamma = -5\mu_\alpha$. Since the system of (3.2) and (3.3) has a non trivial solution, its determinant is zero:

$$(\mu_\alpha + \mu_\gamma)(\mu_\beta + \mu_\gamma) + (\mu_\alpha + \mu_\beta)(\mu_\alpha + \mu_\beta + 2\mu_\gamma) = 0$$

Now, using $\mu_\beta = \mu_\gamma = -5\mu_\alpha$ we derive easily $\mu_\alpha = \mu_\beta = \mu_\gamma = 0$.

Analogously we can see, that

$$g((\nabla_{Z_\beta} J) Z_\beta, Z_\alpha) \neq 0$$

for some α, β implies that M is flat at p .

Suppose, that there exist α, β, γ such that

$$g((\nabla_{Z_\alpha} J) Z_\beta, Z_\gamma) \neq 0.$$

We put in (2.4) $X = Z_\alpha$, $Y = Z_\beta$, $Z = Z_{\bar{\beta}}$, $U = Z_\gamma$, $V = Z_\beta$ and we obtain

$$(3.4) \quad (5\mu_\beta + \mu_\gamma) g((\nabla_{Z_\alpha} J)Z_\beta, Z_\gamma) - (\mu_\alpha + \mu_\beta)g((\nabla_{Z_\beta} J)Z_\alpha, Z_\gamma) = 0 \quad .$$

Let us change here α and β :

$$(3.5) \quad (5\mu_\alpha + \mu_\gamma) g((\nabla_{Z_\beta} J)Z_\alpha, Z_\gamma) - (\mu_\alpha + \mu_\beta)g((\nabla_{Z_\alpha} J)Z_\beta, Z_\gamma) = 0 \quad .$$

Since $g((\nabla_{Z_\alpha} J)Z_\beta, Z_\gamma) \neq 0$ the system (3.4), (3.5) implies

$$(5\mu_\alpha + \mu_\gamma)(5\mu_\beta + \mu_\gamma) = (\mu_\alpha + \mu_\beta)^2 \quad .$$

We can change here β and γ :

$$(3.6) \quad (5\mu_\alpha + \mu_\beta)(5\mu_\gamma + \mu_\beta) = (\mu_\alpha + \mu_\gamma)^2 \quad .$$

Consider the following two possibilities:

Case I. $g((\nabla_{Z_\beta} J)Z_\alpha, Z_\gamma) \neq 0$. Then we have also (changing in (3.6) α and β):

$$(5\mu_\beta + \mu_\alpha)(5\mu_\gamma + \mu_\alpha) = (\mu_\beta + \mu_\gamma)^2 \quad .$$

The system of the last three equations implies $\mu_\alpha = \mu_\beta = \mu_\gamma = 0$.

Case II. $g((\nabla_{Z_\beta} J)Z_\alpha, Z_\gamma) = 0$. Then (3.4), (3.5) imply

$$5\mu_\beta + \mu_\gamma = 0 \quad \text{and} \quad \mu_\alpha + \mu_\beta = 0 \quad ,$$

so $\mu_\gamma = -5\mu_\beta$, $\mu_\alpha = -\mu_\beta$ and using (3.6) we obtain $\mu_\alpha = \mu_\beta = \mu_\gamma = 0$.

Consequently $g((\nabla_{Z_\alpha} J)Z_\beta, Z_\gamma) \neq 0$ always implies $\mu_\alpha = \mu_\beta = \mu_\gamma = 0$. If $n > 3$, we replace in (2.4) $X = Z_\alpha$, $Y = Z_\delta$, $Z = Z_{\bar{\delta}}$, $U = Z_\beta$, $V = Z_\gamma$ for $\delta \neq \alpha, \beta, \gamma$ and we find $\mu_\delta = 0$, so M is flat at p .

At the end, suppose that there exist α, β, γ such that

$$g((\nabla_{Z_{\bar{\alpha}}} J)Z_\beta, Z_\gamma) \neq 0 \quad .$$

We put in (2.4) $X = Z_{\bar{\alpha}}$, $Y = Z_\beta$, $Z = Z_{\bar{\beta}}$, $U = Z_\gamma$, $V = Z_\beta$ to find

$$(5\mu_\beta + \mu_\gamma)g((\nabla_{Z_{\bar{\alpha}}} J)Z_\beta, Z_\gamma) = 0 \quad .$$

Hence $5\mu_\beta + \mu_\gamma = 0$. By the symmetry of β and γ we have also $\mu_\beta + 5\mu_\gamma = 0$ and hence $\mu_\beta = \mu_\gamma = 0$.

Now we put in the second Bianchi identity (2.3) $X = Z_{\bar{\alpha}}, Y = Z_\beta, Z = Z_\gamma, U = Z_\beta, V = Z_{\bar{\beta}}$ and we obtain

$$(\nabla_{Z_\beta} Q)(Z_{\bar{\alpha}}, Z_\gamma) - 2(\nabla_{Z_\gamma} Q)(Z_{\bar{\alpha}}, Z_\beta) = 0 \quad .$$

Hence, because of the symmetry in β and γ we derive easily

$$(3.7) \quad (\nabla_{Z_\beta} Q)(Z_{\bar{\alpha}}, Z_\gamma) = (\nabla_{Z_\gamma} Q)(Z_{\bar{\alpha}}, Z_\beta) = 0 \quad .$$

On the other hand putting in (2.3) $X = Z_\alpha, Y = Z_{\bar{\alpha}}, Z = Z_\beta, U = Z_\gamma, V = Z_{\bar{\alpha}}$ and using (3.7) we obtain:

$$2(\nabla_{Z_{\bar{\alpha}}} Q)(Z_\beta, Z_\gamma) + i\mu_\alpha g((\nabla_{Z_{\bar{\alpha}}} J)Z_\beta, Z_\gamma) = 0 \quad .$$

Since the first term is symmetric in β, γ and the second is antisymmetric, they are both zero, so $\mu_\alpha = 0$.

If $n > 3$ let $\delta \neq \alpha, \beta, \gamma$. We replace in (2.4) $X = Z_{\bar{\alpha}}, Y = Z_\delta, Z = Z_{\bar{\delta}}, U = Z_\beta, V = Z_\gamma$ and we derive easily $\mu_\delta = 0$, so M is flat at p .

On the other hand, if

$$g((\nabla_{Z_\beta} J)Z_\beta, Z_\alpha) \quad g((\nabla_{Z_{\bar{\beta}}} J)Z_\beta, Z_\alpha) \quad g((\nabla_{Z_\alpha} J)Z_\beta, Z_\gamma) \quad g((\nabla_{Z_{\bar{\alpha}}} J)Z_\beta, Z_\gamma)$$

are zero for every α, β, γ , then M is Kählerian at p . This proves our theorem.

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